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q-Moduli of Continuity in $H^p(\mathbb{D})$, $p > 0$, and an Inequality of Hardy and Littlewood

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Some aspects of the interplay between approximation properties of analytic functions and the smoothness of its boundary values are discussed. One main result describes the equivalence of a special q-modulus of continuity and an intrinsic K-functional. Further, a generalization of a theorem due to G. H. Hardy and J. E. Littlewood (1932, *Math. Z.* **34**, 403–439) on the growth of fractional derivatives is deduced with the help of this K-functional.

1. INTRODUCTION

In this paper we discuss some approximation properties of analytic functions on the unit disc $\mathbb{D}$, $f(z)$ from $H^p(\mathbb{D})$, $p > 0$, with finite quasi-norm

$$\|f\|_{H^p} := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\varphi})|^p \, d\varphi \right)^{\frac{1}{p}}, \qquad 0 < p < \infty.$$

In the sequel we will simply write H^p for this space. It is well known that a function $f \in H^p$, $p > 0$, has a nontangential limit $f(e^{it})$ from $L^p(\mathbb{T})$ for

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almost all $t \in [0, 2\pi)$. Here we denote by $L^p(\mathbb{T})$ the measurable, 2π-periodic functions with finite quasi-norm

$$\|f\|_{L^p(\mathbb{T})} \equiv \|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\varphi})|^p \, d\varphi \right)^{\frac{1}{p}}.$$

Then, cf. [19], there holds $\|f\|_{H^p} = \|f\|_p$.

We consider the following two Problems:

(a) the equivalence of a special q-modulus of continuity, based on divided differences, and a "natural" K-functional on H^p,

(b) the characterization of the growth of fractional derivatives of analytic functions on $\mathbb{D}$ by the smoothness of the boundary values.

Concerning Problem (a) we follow Tamrazov [25] and define the q-modulus of continuity $\tilde{\omega}_m(\delta, f)_p$ via divided differences. In our situation these are based on an equidistant partition of the unit circle. The q-modulus has the important property that $\tilde{\omega}_m(\delta, P_{m-1})_p = 0$ for all algebraic polynomials of order $m-1$ (in contrast to the classical modulus of continuity which does not annihilate trigonometric polynomials).

The K-functional in question is given by

$$K_m(\delta, f)_p := \inf_{g^{(m)} \in H^p} \{\|f-g\|_{H^p} + \delta \, \|g^{(m)}\|_{H^p}\}. \tag{1.1}$$

Since $g^{(m)}(z) = (d/dz)^m \, g(z)$ is the standard m-th derivative, this definition of a K-functional on H^p is intrinsic. One can interpret this K-functional as a measure of the approximation of f by an analytic function g with simultaneous control (in norm) of the derivatives of g. This point of view is reflected by the proof of the equivalence between $K_m(\delta, f)_p$ and $\tilde{\omega}_m(\delta, f)_p$ which follows the pattern of a proof developed by Oswald [17] in a related situation. Essential use is made of an inequality of Bernstein–Nikol'skii–Stechkin type and of an inequality of Jackson type, both involving the q-modulus of continuity.

Concerning Problem (b) we recall the following classical result of Hardy and Littlewood [13] on analytic functions on $\mathbb{D}$.

Let $f(z) = \sum a_n z^n$, $f^{\beta}(z) = \sum (\Gamma(n+1)/\Gamma(n+1-\beta)) \, a_n z^{n-\beta}$. *Then* $f(z) \in Lip(\alpha, p)$, $-1+\alpha < \beta < \alpha$, *implies* $f^{\beta}(z) \in Lip(\alpha - \beta, p)$.

This was generalized by Zygmund [28], Brudnyi and Gopengauz [5], Storozenko [23], and others. In view of the above mentioned equivalence, their results may be summarized as

$$\|f^{(m)}(re^{it})\|_p \leqslant C_{m,p}(1-r)^{-m} K_m((1-r)^m, f)_p, \qquad m \in \mathbb{N}. \tag{1.2}$$

We will discuss an extension of (1.2) to appropriate fractional derivatives as well as the corresponding converse inequality (see Theorem 3.1) in Section 3.

2. q-MODULI OF CONTINUITY

The classical modulus of continuity $\omega_m(\delta, f)_p$ is given by

$$\omega_m(\delta, f)_p := \sup_{0<t<\delta} \|\Delta_t^m f(e^{i\varphi})\|_p, \tag{2.1}$$

where

$$\Delta_t^m f(e^{i\varphi}) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(e^{i(\varphi+jt)}).$$

Let us introduce a q-modulus of continuity. The divided difference of a function f with respect to the knots $z_i \in \mathbb{C}$ is given by

$$[z_0, z_1, ..., z_m; f] = \sum_{j=0}^{m} f(z_j) \prod_{i \neq j} (z_j - z_i)^{-1}. \tag{2.2}$$

Choosing $z_j = zq^j$, $z = e^{i\varphi}$, $q = e^{it}$, in (2.2), we define the q-difference operator ∇_q^m by

$$\nabla_q^m f(z) = [z_0, z_1, ..., z_m; f] \prod_{j=1}^{m} (z_j - z_0) \tag{2.3}$$

and, analogously to $\omega_m(\delta, f)_p$, the q-modulus of continuity $\tilde{\omega}_m(\delta, f)_p$ by

$$\tilde{\omega}_m(\delta, f)_p := \sup_{0<t<\delta} \|\nabla_q^m f(e^{it})\|_p. \tag{2.4}$$

We mention the standard properties following from (2.3) (see [6, pp. 120] for the real case and [24])

$$\tilde{\omega}_m(\delta, f+g)_p \lesssim \tilde{\omega}_m(\delta, f)_p + \tilde{\omega}_m(\delta, g)_p, \tag{2.5}$$

$$\tilde{\omega}_m(n\delta, f)_p \lesssim n^{m-1+1/s} \tilde{\omega}_m(\delta, f)_p, \qquad s = \min(1, p), \tag{2.6}$$

$$\tilde{\omega}_m(\delta, f)_p \lesssim \|f\|_p, \qquad \tilde{\omega}_m(\delta, f)_p \lesssim \delta^m \|f^{(m)}\|_p. \tag{2.7}$$

Here and in the following we use the notation $A \lesssim B$ for $A \leqslant C_1 B$ and $A \approx B$ if $C_1 B \leqslant A \leqslant C_2 B$. Positive constants in the estimates of this paper do not depend on f and n.

It is clear from (2.3) and (2.2) that

$$\tilde{\omega}_m(\delta, P_{m-1})_p = 0 \qquad \text{for all algebraic polynomials} \quad P_{m-1}(z) = \sum_{j=0}^{m-1} a_j z^j.$$

Let us also note that

$$\nabla_q^m f(z) = \sum_{j=0}^{m} (-1)^{m-j} q^{-j(m-(j+1)/2)} \begin{bmatrix} m \\ j \end{bmatrix}_q f(zq^j), \tag{2.8}$$

where $[{m \atop j}]_q$ is the polynomial of Gauss

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{(1-q^m)(1-q^{m-1})\ldots(1-q^{m-j+1})}{(1-q^j)(1-q^{j-1})\ldots(1-q)},$$

and that the operator ∇_q^m has the following recurrence property (see [10, 23])

$$\nabla_q^0 f(z) = f(z), \qquad \nabla_q^m f(z) = q^{-(m-1)} \nabla_q^{m-1} f(zq) - \nabla_q^{m-1} f(z), \qquad m \in \mathbb{N}. \tag{2.9}$$

We want to identify $\tilde{\omega}_m$ with the above K-functional. To this end we need analogs of the useful relations $\Delta_t^m \Delta_t^k f = \Delta_t^{m+k} f$, $(\Delta_t^m f)' = \Delta_t^m f'$. For this purpose we introduce generalized finite differences.

2.1. *Generalized Finite Differences*

Let $f(z)$ be defined in points zq^j, $j = 0, ..., m$, where $z, q \in \mathbb{C}$ are arbitrary. We define generalized finite differences $\nabla_q^{m,s}$, $m \in \mathbb{N}$, $s \in \mathbf{Z}$, by the following recurrence formulas

$$\nabla_q^{0,s} f(z) = f(z), \qquad \nabla_q^{m,s} f(z) = q^{-(m-1+s)} \nabla_q^{m-1,s} f(zq) - \nabla_q^{m-1,s} f(z). \tag{2.10}$$

Clearly, by (2.9), $\nabla_q^{m,0} f(z) = \nabla_q^m f(z)$. Generalized divided differences $\nabla_q^{m,s}$ have some useful properties and we shall establish these properties in the next lemma. The most interesting ones are (2.13) and (2.14).

LEMMA 2.1. *Let* $z, q \in \mathbb{C}$ *and* $\nabla_q^{m,s}$ *be defined by* (2.10). *Then, for all* $m \in \mathbb{N}$, $s \in \mathbf{Z}$, *and appropriate* f

$$\begin{aligned}(-1)^{m+1} & \nabla_q^{m+1,s} f(z) \\ &= \sum_{k_1=0}^{1} \cdots \sum_{k_{m+1}=0}^{1} (-1)^{\sum_{j=1}^{m+1} k_j} q^{-\sum_{j=1}^{m+1} (j-1+s) k_j} f(zq^{\sum_{j=1}^{m+1} k_j});\end{aligned} \tag{2.11}$$

$$\nabla_q^{m+1,\,s-1} f(z) = q^{-(s-1)} \nabla_q^{m,\,s} f(zq) - \nabla_q^{m,\,s} f(z); \tag{2.12}$$

$$\nabla_q^{s,\,0} \nabla_q^{m,\,s} f(z) = \nabla_q^{m+s,\,0} f(z); \tag{2.13}$$

$$(\nabla_q^{m,\,s} f(z))' = \nabla_q^{m,\,s-1} f'(z). \tag{2.14}$$

Proof. The identities of Lemma 2.1 will be established by induction on m. The proof of the first and the second identity is immediate. For proving the third one we shall use (2.12). The identity (2.14) is a consequence of (2.11). The direct calculations are given below.

For $m=0$ the identities (2.11)–(2.13) are obvious. Assume now that (2.11)–(2.13) hold for $m-1$. We will prove that they are also true for m.

(1) Concerning (2.11) we have

$$\begin{aligned}
\nabla_q^{m+1,\,s} f(z) &= (-1)^m \sum_{k_1=0}^{1} \cdots \sum_{k_m=0}^{1} (-1)^{\sum_{j=1}^m k_j} q^{-\sum_{j=1}^m (j-1+s)\,k_j} \\
&\quad \times \{ f(zq^{\sum_{j=1}^m k_j + 1}) \cdot q^{-m-s} - f(zq^{\sum_{j=1}^m k_j + 0}) \cdot q^{(-m-s)\cdot 0} \} \\
&= (-1)^{m+1} \sum_{k_1=0}^{1} \cdots \sum_{k_{m+1}=0}^{1} (-1)^{\sum_{j=1}^{m+1} k_j} q^{-\sum_{j=1}^{m+1} (j-1+s)\,k_j} f(zq^{\sum_{j=1}^{m+1} k_j}).
\end{aligned}$$

(2) Analogously, from (2.10), we obtain

$$\begin{aligned}
\nabla_q^{m+1,\,s-1} f(z) &= q^{-(m+s-1)} \nabla_q^{m,\,s-1} f(zq) - \nabla_q^{m,\,s-1} f(z) \\
&= q^{-(m+s-1)} \{ q^{-(s-1)} \nabla_q^{m-1,\,s} f(zq^2) - \nabla_q^{m-1,\,s} f(zq) \} \\
&\quad - \{ q^{-(s-1)} \nabla_q^{m-1,\,s} f(zq) - \nabla_q^{m-1,\,s} f(z) \} \\
&= q^{-(s-1)} \nabla_q^{m,\,s} f(zq) - \nabla_q^{m,\,s} f(z),
\end{aligned}$$

which proves (2.12).

(3) Finally

$$\begin{aligned}
\nabla_q^{s,\,0} \nabla_q^{m,\,s} f(z) &= \nabla_q^{s,\,0} (q^{-s} \nabla_q^{m-1,\,s+1} f(zq) - \nabla_q^{m-1,\,s+1} f(z)) \\
&= q^{-s} \nabla_q^{s,\,0} \nabla_q^{m-1,\,s+1} f(zq) - \nabla_q^{s,\,0} \nabla_q^{m-1,\,s+1} f(z) \\
&= \nabla_q^{s+1,\,0} \nabla_q^{m-1,\,s+1} f(z) = \nabla_q^{m+s,\,0} f(z).
\end{aligned}$$

2.2. *Inequality of Bernstein–Nikolskii–Stechkin*

Now we shall use the properties of the generalized differences to prove the following analog of the classical inequality.

THEOREM 2.2. *For algebraic polynomials P_n of order at most n there holds*

$$\|P_n^{(m)}\|_p \lesssim n^m \|\nabla_q^m P_n\|_p, \qquad q = e^{i/n}.$$

Proof. For $m = 1$ the estimate

$$\|P_n'\|_p \lesssim n \|\nabla_q^1 P_n\|_p = n \|\Delta_{1/n}^1 P_n\|_p \tag{2.15}$$

seems to be known and can be obtained by using the multiplier theorem for H^p (see [2, 21]). By this inequality and the properties of $\nabla_q^{m,s}$ we have, using (2.13) and (2.14) repeatedly,

$$\begin{aligned}
\|P_n^{(m)}\|_p &\lesssim (n-m+1) \|\nabla_q^1 P_n^{(m-1)}\|_p \\
&= (n-m+1) \|(\nabla_q^{1,1} P_n^{(m-2)})'\|_p \\
&\lesssim (n-m+1)(n-m+2) \|\nabla_q^{1,0} \nabla_q^{1,1} P_n^{(m-2)}\|_p \\
&= (n-m+1)(n-m+2) \|\nabla_q^2 P_n^{(m-2)}\|_p \lesssim \cdots \\
&\lesssim n(n-1)\ldots(n-m+1) \|\nabla_q^m P_n\|_p \lesssim n^m \|\nabla_q^m P_n\|_p.
\end{aligned}$$

2.3. *Jackson's Theorem in H^p*

The following theorem was proved by Storozenko [22].

THEOREM A. *Given $f \in H^p$, $0 < p < \infty$, and $m \in \mathbb{N}$, there is a polynomial Q_n of degree $n \in \mathbb{N}$ such that*

$$\|f - Q_n\|_{H^p} \lesssim \omega_m(n^{-1}, f)_p.$$

We shall prove the following variant.

THEOREM 2.3. *Let $f \in H^p$, $0 < p < \infty$, and $m \in \mathbb{N}$ be given. Then there is a polynomial R_n of degree $n > m$ such that*

$$\|f - R_n\|_{H^p} \lesssim \tilde{\omega}_m(n^{-1}, f)_p.$$

Proof. Let $q = re^{it}$, $0 < r < 1$, $\alpha > 0$,

$$K_n^\alpha(q) = (A_n^\alpha)^{-1} \frac{q^{-n}}{(1-q)^{1+\alpha}}, \qquad A_n^\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{q^{-n}}{(1-q)^{1+\alpha}}\, dt = \binom{\alpha+n}{n}.$$

Note that the functions $K_n^\alpha(q)$ are the kernels of the (C, α)-means of functions which are analytic in $\mathbb{D}$. With the help of these kernels write

$$R_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(z) + (-1)^{m-1} \nabla_q^m f(z)\} K_{n-m+1}^\alpha(q)\, dt$$

$$= \frac{1}{2\pi} \sum_{k=1}^{m} (-1)^{k-1} \int_{-\pi}^{\pi} q^{-k(m-(k+1)/2)} \begin{bmatrix} m \\ k \end{bmatrix}_q f(zq^k)\, K_{n-m+1}^\alpha(q)\, dt.$$

Then $R_n(z) = R_n(z; f, m, \alpha)$ is independent of r and is an algebraic polynomial of degree at most n. Indeed, with

$$f(u) = \sum_{j=0}^{\infty} a_j u^j, \qquad (1-u)^{-(1+\alpha)} = \sum_{l=0}^{\infty} b_l(\alpha)\, u^l, \qquad |u| < 1,$$

we have

$$f(u_1)(1-u_2)^{-(1+\alpha)} = \sum_{p=0}^{\infty} \sum_{l+j=p} a_j b_l(\alpha)\, u_1^j u_2^l.$$

Putting $u_1 = zq^k$ and $u_2 = q$, we obtain

$$q^{-k(m-(k+1)/2)} \begin{bmatrix} m \\ k \end{bmatrix}_q f(zq^k)(1-q)^{-(1+\alpha)}$$

$$= \sum_{p=0}^{\infty} \sum_{l+j=p} a_j b_l(\alpha)\, z^j q^{jk+l} q^{-k(m-(k+1)/2)} \begin{bmatrix} m \\ k \end{bmatrix}_q. \tag{2.16}$$

The polynomial $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_q = \sum c_s q^s$ is a polynomial in q of degree at most $k(m-k)$. After multiplying (2.16) by q^{-n+m-1} and integrating it with respect to t from $-\pi$ to π there remain only such terms in the resulting double sum that satisfy

$$jk + l - k(m-(k+1)/2) + s - n + m - 1$$
$$= k(j-m+1) - (n-m+1) + k(k+1)/2 + l + s = 0.$$

From the last equality it follows that $j \leqslant n$, so $R_n(z)$ is a polynomial of degree at most n.

Suppose first that $0 < p < 1$. Fix $m \in \mathbb{N}$, $n > m$, and choose α in such a way that

$$2 \geqslant (1+\alpha)\, p - (m-1)\, p - 1 > 1 \qquad \text{or} \qquad \alpha \in (2p^{-1} + m - 2,\, 3p^{-1} + m - 2]$$

is satisfied. We prove the assertion of Theorem 2.3 with $R_n(z) = R_n(z; f, k, \alpha)$. We shall use the following facts for H^p-functions. There holds

$$\int_{-\pi}^{\pi} f(q)\,\frac{q^{-n}}{(1-q)^{1+\alpha}}\,dt = \int_{-\pi}^{\pi} f(q)\,q^{-n}\left(\frac{1-q^{n+1}}{1-q}\right)^{1+\alpha} dt$$

for $n \in \mathbb{N}$, $\alpha > 0$, and

$$\left(\int_{-\pi}^{\pi} |f(q)|\,dt\right)^p \lesssim (1-r)^{p-1} \int_{-\pi}^{\pi} |f(q/r)|^p\,dt, \qquad 0 < p, r < 1. \tag{2.17}$$

The first fact follows by the orthogonality of the functions $q^k = r^k e^{ikt}$. For the second see, for example, [9, Chap. 2, Exercises]. Now, put $n_1 = n - m + 1$ and estimate

$$\begin{aligned}
&|f(e^{i\varphi}) - R_n(e^{i\varphi})|^p\\
&\quad \leqslant \left(|q|^{-m(m-1)/2} \int_{-\pi}^{\pi} |q^{m(m-1)/2}\nabla_q^m f(e^{i\varphi})\,K_{n_1}^{\alpha}(q)|\,dt\right)^p\\
&\quad \lesssim (1-r)^{p-1} A_{n_1}^{-\alpha p} r^{-p(n+m(m-1)/2)} \int_{-\pi}^{\pi} \left|\nabla_{q/r}^m f(e^{i\varphi}) \left(\frac{1-(q/r)^{n_1}}{1-q/r}\right)^{1+\alpha}\right|^p dt.
\end{aligned}$$

Choose $r = 1 - n_1^{-1}$. Then it follows that $(1-r)^{p-1} \approx n_1^{1-p}$, $A_{n_1}^{-\alpha p} \approx n_1^{-\alpha p}$, and $r^{-p(n+m(m-1)/2)} \approx 1$ with constants independent of n_1. An integration of the last inequality leads to

$$\|f - R_n\|_p^p \lesssim n_1^{1-(\alpha+1)p} \int_0^{\pi} \tilde{\omega}_m^p(t, f)_p \left|\frac{\sin(n_1 t/2)}{\sin(t/2)}\right|^{(1+\alpha)p} dt =: n_1^{1-(\alpha+1)p} I.$$

Decompose the integral I at the level $1/n_1$ in two integrals and apply to the second one (2.6) to obtain

$$\begin{aligned}
I &\lesssim \tilde{\omega}_m^p\left(\frac{1}{n_1}, f\right)_p \left\{n_1^{(\alpha+1)p} \int_0^{1/n_1} dt + n_1^{(m-1)p+1} \int_{1/n_1}^{\pi} t^{-(1+\alpha)p+(m-1)p+1}\,dt\right\}\\
&\lesssim n_1^{(\alpha+1)p-1} \tilde{\omega}_m^p\left(\frac{1}{n_1}, f\right)_p.
\end{aligned}$$

Summarizing, for $0 < p < 1$ we have

$$\|f - R_n\|_p \lesssim \tilde{\omega}_m\left(\frac{1}{n}, f\right)_p.$$

Now assume $p \geqslant 1$ and choose $\alpha > m$. Then, by the integral Minkowski inequality,

$$\begin{aligned}
\|f-R_n\|_p &\leqslant (A_{n_1}^{\alpha})^{-1} \int_{-\pi}^{\pi} \|\nabla_{q/r}^m f\|_p \left|\frac{\sin(n_1 t/2)}{\sin(t/2)}\right|^{1+\alpha} dt \\
&\lesssim n_1^{-\alpha} \tilde{\omega}_m\left(\frac{1}{n_1}, f\right)_p \left(\int_0^{1/n_1} \left|\frac{\sin(n_1 t/2)}{\sin(t/2)}\right|^{1+\alpha} dt + n_1^m \int_{1/n_1}^{\pi} t^{m-\alpha-1} dt\right) \\
&\lesssim \tilde{\omega}_m\left(\frac{1}{n_1}, f\right)_p \lesssim \tilde{\omega}_m\left(\frac{1}{n}, f\right)_p.
\end{aligned}$$

2.4. *K-Functional and q-Moduli of Continuity*

The equivalence of the modulus of continuity $\tilde{\omega}_m$ and the K_m-functional in H^p (see (2.4) and (1.1)) will now be deduced with the aid of Theorems 2.2 and 2.3.

THEOREM 2.4. *For $f \in H^p$, $0 < p < \infty$, $m \in \mathbb{N}$, $0 < \delta < \pi$, there holds*

$$\tilde{\omega}_m(\delta, f)_p \approx K_m(\delta^m, f)_p.$$

Proof. The inequality $\tilde{\omega}_m(\delta, f)_p \lesssim K_m(\delta^m, f)_p$ follows from (2.7) ($s = \min\{1, p\}$)

$$\tilde{\omega}_m^s(\delta, f)_p \leqslant \tilde{\omega}_m^s(\delta, f-g)_p + \tilde{\omega}_m^s(\delta, g)_p \lesssim \{\|f-g\|_p^s + \delta^{ms} \|g^{(m)}\|_p^s\}.$$

The converse estimate

$$K_m(\delta^m, f)_p \lesssim \tilde{\omega}_m(\delta, f)_p$$

turns out to be a consequence of Theorem 2.2 and Theorem 2.3.

For proving this, first suppose that $0 < \delta \leqslant m^{-1}$ and choose n such that $n^{-1} < \delta \leqslant (n-1)^{-1}$. With the polynomial R_n from Theorem 2.3 it follows by Theorem 2.2 (with $q = e^{i/n}$), Theorem 2.3 and (2.7) that

$$\begin{aligned}
K_m(\delta^m, f)_p &\leqslant \|f-R_n\|_p + n^{-m} \|R_n^{(m)}\|_p \\
&\lesssim (\tilde{\omega}_m(n^{-1}, f)_p + \|\nabla_q^m R_n\|_p) \\
&\lesssim (\tilde{\omega}_m(\delta, f)_p + \|\nabla_q^m (R_n - f)\|_p + \|\nabla_q^m f\|_p) \\
&\lesssim \tilde{\omega}_m(\delta, f)_p.
\end{aligned}$$

But this estimate also holds for $m^{-1} < \delta < \pi$, since by (2.6)

$$K_m(\delta^m, f)_p \leqslant K_m(\pi^m, f)_p \leqslant (\pi m)^m K_m((1/m)^m, f)_p$$
$$\lesssim \tilde{\omega}_m(1/m, f)_p \leqslant \tilde{\omega}_m(\delta, f)_p.$$

This characterization of the q-modulus of continuity allows to give an improvement of (2.6) for functions from H^p in the case $0 < p < 1$.

COROLLARY 2.5. *If $f \in H^p$, $0 < p < \infty$, $m, n \in \mathbb{N}$, then for $0 \leqslant \delta \leqslant \pi n^{-1}$*

$$\tilde{\omega}_m(n\delta, f)_p \lesssim n^m \tilde{\omega}_m(\delta, f)_p. \tag{2.18}$$

Proof. By Theorem 2.4,

$$\tilde{\omega}_m(n\delta, f)_p \approx \inf_{g^{(m)} \in H^p} \{\|f-g\|_p + (n\delta)^m \|g^{(m)}\|_p\}$$
$$\lesssim n^m \inf_{g^{(m)} \in H^p} \{\|f-g\|_p + \delta^m \|g^{(m)}\|_p\} \approx n^m \tilde{\omega}_m(\delta, f)_p.$$

Remark 2.6. A result of Oswald [17] (see also [2]) states the following.

If $f \in H^p$, $0 < p < \infty$, $m \in \mathbb{N}$, $0 < \delta \leqslant \pi$, then (see (2.1)*)*

$$\omega_m(\delta, f)_p \approx \inf_{\frac{\partial^m g}{\partial t^m} \in H^p} \left\{\|f-g\|_p + \delta^m \left\|\frac{\partial^m g}{\partial t^m}\right\|_p\right\}. \tag{2.19}$$

We want to compare ω_m and $\tilde{\omega}_m$. For this purpose we prove

LEMMA 2.7. *Let $f^{(m)} \in H^p$, $0 < p < \infty$, $m \in \mathbb{N}$, and $f_m(z) = f(z) - \sum_{j=0}^{m-1} f^{(j)}(0)\, z^j/j!$. Then*

$$\|f_m^{(m)}\|_p \approx \left\|\frac{\partial^m f_m}{\partial t^m}\right\|_p. \tag{2.20}$$

Proof. It is evident by the identity $(\partial f(re^{it}))/(\partial t) = izf'(z)$, $z = e^{it}$, that $\|\partial f/\partial t\|_p = \|f'\|_p$. Suppose now that $f(0) = 0$, $f' \in H^p$. By the maximal theorem of Hardy and Littlewood [12] (see also [9]), we obtain

$$\|f\|_p = \left\|f(0) + \int_0^1 f'(re^{it})\, e^{it}\, dr\right\|_p \leqslant \|\sup_{0<r<1} |f'(re^{it})|\,\|_p \lesssim \|f'\|_p = \left\|\frac{\partial f}{\partial t}\right\|_p.$$

From this inequality we have

$$\left\|\frac{\partial^j f_m}{\partial t^j}\right\|_p \lesssim \left\|\frac{\partial^m f_m}{\partial^m t}\right\|_p, \qquad j = 1, ..., m-1. \tag{2.21}$$

By the identity

$$f^{(m)}(z)=z^{-m}\sum_{j=1}^{m} b_j \frac{\partial^j f(z)}{\partial t^j}, \qquad z\neq 0, \quad b_j\in\mathbb{C}, \quad j=1, ..., m$$

and by (2.21) we obtain

$$\|f_m^{(m)}\|_p \lesssim \sum_{j=1}^{m} \left\|\frac{\partial^j f_m}{\partial t^j}\right\|_p \lesssim \left\|\frac{\partial^m f_m}{\partial t^m}\right\|_p.$$

The converse estimate can be deduced in the same way.

A combination of Theorem 2.4, Lemma 2.7, and (2.19) gives the following result.

THEOREM 2.8. *If $f\in H^p$, $0<p<\infty$, then*

$$\tilde{\omega}_m(\delta, f)_p=\tilde{\omega}_m(\delta, f_m)_p\approx\omega_m(\delta, f_m)_p, \qquad m\in\mathbb{N}, \quad 0<\delta<\pi. \tag{2.22}$$

3. A HARDY–LITTLEWOOD TYPE THEOREM FOR FRACTIONAL DERIVATIVES

Let $f(z)=\sum_{j=0}^{\infty}\hat{f}(j)\, z^j$ be an analytic function on the unit disk $\mathbb{D}$. For positive α, define fractional derivatives in sense of Riemann–Liouville by

$$f^{(\alpha)}(z)=\sum_{j=[\alpha]}^{\infty}\frac{\Gamma(j-[\alpha]+1+\alpha)}{\Gamma(j-[\alpha]+1)}\hat{f}(j)\, z^{j-[\alpha]}, \tag{3.1}$$

where $[\alpha]=\max\{i\leqslant\alpha : i\in\mathbb{N}_0\}$. If $\alpha=k\in\mathbb{N}$, then $f^{(k)}$ is the usual derivative. This definition can be found in the work of Pekarskii [18]; it is in particular placed in the circle of related fractional derivatives in [20, Subsect. 23.2]. Analogously to (1.1), we define K-functionals with respect to fractional derivatives of order α by

$$K_\alpha(\delta, f)_p=\inf_{g^{(\alpha)}\in H^p}\{\|f-g\|_p+\delta\,\|g^{(\alpha)}\|_p\}.$$

3.1. *A Hardy–Littlewood Type Theorem*

The main result of this section is the following theorem.

THEOREM 3.1. *Let f be an analytic function on the unit disc; let $0<p<\infty$ and $\alpha>0$.*

(A) *If $f \in H^p$, then*

$$\|f^{(\alpha)}(re^{it})\|_p \lesssim (1-r)^{-\alpha} K_\alpha((1-r)^\alpha, f)_p, \qquad 0<r<1. \tag{3.2}$$

(B) *Let $\omega(t)$ be a nondecreasing, continuous function on $[0, 1]$ with $\omega(0)=0$ and*

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \lesssim \omega(\delta). \tag{3.3}$$

Then,

$$\|f^{(\alpha)}(re^{it})\|_p \lesssim (1-r)^{-\alpha}\, \omega(1-r), \qquad r \to 1-, \tag{3.4}$$

implies

$$f \in H^p \qquad \text{and} \qquad K_\alpha(\delta^\alpha, f)_p \lesssim \omega(\delta). \tag{3.5}$$

Remarks 3.2. (i) Note, that part B of Theorem 3.1 for natural α is contained in [27]; there, condition (3.3) was given in an equivalent, but complicated form. An elementary proof can be found in [15].

(ii) Let us mention that (3.3) is equivalent to

$$\int_0^\delta t^{-1}\omega^q(t)\, dt \lesssim \omega^q(\delta), \qquad q>0. \tag{3.6}$$

To see this, denote the inequality (3.6) by A_q. Then, on the one hand, it follows from the monotonicity of ω that A_q implies $A_{q+\varepsilon}$ for all $\varepsilon>0$. On the other hand, a multiplication of the Dini condition A_q by δ^{-1} and an integration with respect to δ (say from 0 to δ_1) lead to

$$\int_0^\delta t^{-1}\omega^q(t) \log(\delta/t)\, dt \lesssim \omega^q(\delta).$$

Now an application of Hölder's inequality from below (with $0<r=(q-\varepsilon)/q<1$) shows that A_q implies $A_{q-\varepsilon}$ for $0<\varepsilon<q/2$.

(iii) It is a well known fact that the condition $\omega_k(\delta, f)_p = O(\delta^\alpha)$, $k>\alpha$, is not sufficient for $f^{(\alpha)} \in H^p$, $p>0$. Moreover, the results of Hardy and Littlewood [11, 13] imply that

$$f^{(\alpha)} \in H^p \Rightarrow \omega_1(\delta, f)_p = o(\delta^\alpha), \qquad 0<\alpha<1, p \geqslant 1.$$

Theorem 3.1 gives the following refinement.

COROLLARY 3.3. *For* $\alpha, p > 0$ *the following conditions are equivalent:*

(1) $f^{(\alpha)} \in H^p$;

(2) $K_\alpha(\delta^\alpha, f)_p = O(\delta^\alpha)$.

In combination with the results in [26], which can be extended to H^p, Corollary 3.3 contains a further characterization of well known saturation classes in approximation theory.

COROLLARY 3.4. *For* $\alpha, p > 0$ *and* $P_n(z) = \sum_{k=0}^{n} a_k z^k$ *we have the following Bernstein–Nikolskii–Stechkin type inequality*

$$\|P_n^{(\alpha)}\|_p \lesssim n^\alpha K_\alpha(n^{-\alpha}, P_n)_p. \tag{3.7}$$

This follows by Lemma C below and Theorem 3.1, part A.

The proof of Theorem 3.1 is split up into a series of lemmas.

LEMMA B [13, 15]. *For* $f \in H^p$, $0 < p < \infty$, $m \in \mathbb{N}$, *there holds*

$$\|f^{(m)}(re^{it})\|_p \lesssim (1-r)^{-m} \|f\|_p, \qquad 0 < r < 1.$$

LEMMA C. *Let* $P_n(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial and* $n \geqslant 0$, $p > 0$. *Then*

$$\|P_n(e^{it})\|_p \leqslant e\, \|P_n((1-1/(n+1))\, e^{it})\|_p. \tag{3.8}$$

Proof. This is a variant of one of the main tools in [14]. For the sake of completeness we prove it.

In the case $n = 0$ the inequality (3.8) is obvious. Let $n \geqslant 1$. First note that for functions $F(z)$, analytic on $\{z: |z| > r_0 > 0\} \cup \{\infty\}$, we have

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(Re^{it})|^p\, dt\right)^{1/p} \leqslant \left(\int_{-\pi}^{\pi} |F(re^{it})|^p\, dt\right)^{1/p} \tag{3.9}$$

provided $0 < r_0 < r \leqslant R < \infty$, $0 < p < \infty$. Consider now the function $F_1(z) = P_n(z)\, z^{-n}$. It is clear that $F_1(z)$ is analytic in the domain $\{z: |z| > r_0\} \cup \{\infty\}$, $r_0 > 0$. From (3.9) it follows that

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |P_n(e^{it})|^p\, dt\right)^{1/p}$$
$$\leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} (1-1/(n+1))^{-np}\, |P_n((1-1/(n+1))\, e^{it})|^p\, dt\right)^{1/p}.$$

LEMMA D [13]. *If* $f \in H^p$, $0 < p < \infty$, $s = \min(1, p)$, $\alpha > 0, 0 < r < 1$, *then*

$$\left\| \int_r^1 (1-\rho)^{\alpha-1} |f(\rho e^{it})| \, d\rho \right\|_p^s \lesssim \int_r^1 (1-\rho)^{\alpha s-1} \|f(\rho e^{it})\|_p^s \, d\rho. \tag{3.10}$$

The following variant of the definition (3.1) of a fractional derivative is due to Flett [8]

$$J^\alpha f(z) = \sum_{j=0}^{\infty} (j+1)^\alpha \hat{f}(j) \, z^j, \qquad J^k f(z) = \left[\frac{d}{dz} z \right]^k f(z) \qquad \text{if} \quad k \in \mathbb{N}. \tag{3.11}$$

It has the useful semi-group property $J^{\alpha+\beta} = J^\alpha J^\beta$. It is related to the fractional derivative $f^{(\alpha)}$, given by (3.1), in the following way (for the equivalence in (3.12) see [18, Lemma 1]),

$$\|f^{(\alpha)}\|_p \approx \|J^\alpha f_{[\alpha]}\|_p \lesssim \|J^\alpha f\|_p + \sum_{k=0}^{[\alpha]-1} (k+1)^\alpha \, |\hat{f}(k)| \lesssim \|J^\alpha f\|_p, \tag{3.12}$$

where

$$f_{[\alpha]}(z) = \sum_{k=[\alpha]}^{\infty} \hat{f}(k) \, z^k \tag{3.13}$$

since, by [7, p. 98], $(k+1)^\alpha \, |\hat{f}(k)| = |\widehat{J^\alpha f}(k)| \lesssim \|J^\alpha f\|_p$, $0 \leqslant k \leqslant [\alpha]-1$.

The next lemma is a consequence of Lemmas B and C.

LEMMA 3.5. *Let* $f \in H^p$, $0 < p < \infty$, $\alpha > 0$. *Then*

$$\|J^\alpha f(re^{it})\|_p \lesssim (1-r)^{-\alpha} \, \|f\|_p, \qquad 0 < r < 1. \tag{3.14}$$

Proof. First note that for natural $\alpha = m$ the lemma follows directly from Lemma B. Indeed, $J^m f(z) = \sum_{j=0}^m c_j z^j f^{(j)}(z)$ and, for $s = \min(1, p)$,

$$\|J^m f(re^{it})\|_p^s \lesssim \sum_{j=0}^m \|f^{(j)}(re^{it})\|_p^s \lesssim (1-r)^{-ms} \, \|f\|_p^s. \tag{3.15}$$

Let now $\alpha \neq m \in \mathbb{N}$; set $m = [\alpha]+1$ and $\beta = m - \alpha$. Then there holds the integral representation

$$J^\alpha f(re^{it}) = \frac{1}{\Gamma(\beta)} \int_0^1 \left(\log \frac{1}{\rho} \right)^{\beta-1} J^m f(\rho r e^{it}) \, d\rho, \qquad 0 < r < 1, \tag{3.16}$$

which is due to Flett [8]. From the estimates (3.16), (3.15), (3.10) we conclude that

$$\begin{aligned}\|J^\alpha f(re^{it})\|_p^s &\lesssim \int_0^1 (1-\rho)^{\beta s-1}\,\|J^m f(\rho re^{it})\|_p^s\,d\rho \\ &\lesssim \|f\|_p^s \int_0^1 (1-\rho)^{\beta s-1}\,(1-\rho r)^{-ms}\,d\rho \\ &\lesssim (1-r)^{-\alpha s}\,\|f\|_p^s \int_1^{(1-r)^{-1}} (t-1)^{\beta s-1}\,t^{-ms}\,dt \\ &\lesssim (1-r)^{-\alpha s}\,\|f\|_p^s,\end{aligned}$$

where we used the substitution $t=(1-\rho r)/(1-r)$.

Lemma 3.6. *Let $\alpha, \beta \geqslant 0$ and let $g^{(\beta)} \in H^p$. Then*

$$\|g^{(\alpha+\beta)}((1-\delta)\,e^{it})\|_p \lesssim \delta^{-\alpha}\,\|g^{(\beta)}(e^{it})\|_p, \qquad 0<\delta<1.$$

Proof. By (3.12), the semi-group property of J^α and by Lemma 3.5 we have

$$\|g^{(\alpha+\beta)}((1-\delta)\,e^{it})\|_p \lesssim \|J^{(\alpha+\beta)}g((1-\delta)\,e^{it})\|_p \lesssim \delta^{-\alpha}\,\|J^\beta g\|_p.$$

On account of the definition of $g^{(\beta)}$ we may assume without loss of generality that $g^{(j)}(0)=0$ for $j=0, ..., [\beta]-1$, hence, by (3.12), $\|J^\beta g\|_p \lesssim \|g^{(\beta)}\|_p$ and the assertion is established.

Proof of Theorem 3.1. By Lemma 3.6 ($\beta=0$) it follows for arbitrary $g^{(\alpha)} \in H^p$ that

$$\begin{aligned}(1-r)^\alpha\,\|f^{(\alpha)}(re^{it})\|_p &\lesssim (1-r)^\alpha\,\{\|f^{(\alpha)}(re^{it})-g^{(\alpha)}(re^{it})\|_p+\|g^{(\alpha)}(re^{it})\|_p\} \\ &\lesssim \{\|f-g\|_p+(1-r)^\alpha\,\|g^{(\alpha)}\|_p\}.\end{aligned}$$

Thus part (A) of Theorem 3.1 is proved.

The assertion of part (B) is immediate by the hypotheses (3.4), (3.3) together with the following lemma.

Lemma 3.7. *Let $f \in H^p$, $p>0$, $s=\min(1, p)$, $\alpha>0$, and $0<\delta<1/([\alpha]+2)$. Then*

$$K_\alpha(\delta^\alpha, f)_p^s \lesssim \int_{1-\delta}^1 (1-\rho)^{\alpha s-1}\,\|f^{(\alpha)}(\rho e^{it})\|_p^s\,d\rho.$$

Proof. By the definition of the fractional derivative we have $P^{(\alpha)}_{[\alpha]-1}(z)=0$ and hence $K_\alpha(\delta^\alpha, P_{[\alpha]-1})=0$. Therefore, we may assume without loss of generality

$$f^{(j)}(0)=0, \qquad j=0, ..., [\alpha]-1. \tag{3.17}$$

Let $m=[\alpha]+1$. First we prove the estimate

$$K_\alpha(\delta^\alpha, f)_p \lesssim \tilde{\omega}_m(\delta, f)_p+\delta^\alpha \|f^{(\alpha)}((1-\delta)\, e^{it})\|_p. \tag{3.18}$$

Choose $n \in \mathbb{N}, n>m+1$, such that $(n+1)^{-1}<\delta \leqslant n^{-1}$ and R_n from Theorem 2.3 to obtain

$$K_\alpha(\delta^\alpha, f)_p \leqslant \|f-R_n\|_p+n^{-\alpha} \|R_n^{(\alpha)}\|_p.$$

Now, by Lemma C,

$$\begin{aligned} n^{-\alpha} \|R_n^{(\alpha)}\|_p &\lesssim n^{-\alpha} \|R_n^{(\alpha)}((1-n^{-1})\, e^{it})\|_p \\ &\lesssim \delta^\alpha(\|R_n^{(\alpha)}((1-\delta)\, e^{it})-f^{(\alpha)}((1-\delta)\, e^{it})\|_p+\|f^{(\alpha)}((1-\delta)\, e^{it})\|_p). \end{aligned}$$

Lemma 3.6 and Theorem 2.3 imply (3.18). Since

$$\delta^{s\alpha} \|f^{(\alpha)}((1-\delta)\, e^{it})\|_p^s \lesssim \int_{1-\delta}^1 (1-\rho)^{\alpha s-1} \|f^{(\alpha)}(\rho e^{it})\|_p^s \, d\rho$$

it is sufficient to prove that

$$K_m(\delta^m, f)_p^s \lesssim \int_{1-\delta}^1 (1-\rho)^{\alpha s-1} \|f^{(\alpha)}(\rho e^{it})\|_p^s \, d\rho.$$

For this purpose consider the Taylor expansion of f.

$$f_r(z)=\sum_{j=0}^{m-1} \frac{f^{(j)}(rz)}{j!} (z-rz)^j, \qquad r=1-\delta.$$

It is clear that

$$f(z)-f_r(z)=\frac{1}{(m-1)!} \int_{rz}^z f^{(m)}(\zeta)(z-\zeta)^{m-1} \, d\zeta.$$

Hence

$$\begin{aligned} K_m((1-r)^m, f)_p &\leqslant \|f-f_r\|_p+(1-r)^m \|f_r^{(m)}\|_p \\ &\lesssim \left\| \int_{rz}^z f^{(m)}(\zeta)(z-\zeta)^{m-1} \, d\zeta \right\|_p+\sum_{j=0}^{m-1} \|f^{(j+m)}(rz)(1-r)^{j+m}\|_p. \end{aligned}$$

By (3.10) we obtain

$$\left\| \int_{rz}^{z} f^{(m)}(\zeta)(z-\zeta)^{m-1}\, d\zeta \right\|_p^s \lesssim \left\| \int_r^1 |f^{(m)}(\rho e^{i\varphi})|\, (1-\rho)^{m-1}\, d\rho \right\|_p^s$$

$$\lesssim \int_r^1 (1-\rho)^{ms-1}\, \|f^{(m)}(\rho e^{i\varphi})\|_p^s\, d\rho =: I_m.$$

Now Lemma 3.7 follows since by Lemma 3.6

$$I_m \lesssim \int_r^1 (1-\rho)^{ms-1}\, ((1-\rho)/2)^{s(\alpha-m)}\, \|f^{(\alpha)}((1+\rho)/2\, e^{i\varphi})\|_p^s\, d\rho$$

$$\lesssim \int_r^1 (1-\rho)^{\alpha s-1}\, \|f^{(\alpha)}(\rho\, e^{i\varphi})\|_p^s\, d\rho.$$

Remark 3.8. Lemma C and Lemma 3.5 also give the following inequality of Bernstein-type for fractional derivatives

$$\|J^{\alpha} P_n\|_p \leqslant C(\alpha, p)(n+1)^{\alpha}\, \|P_n\|_p, \qquad \alpha > 0, \quad p > 0.$$

This inequality is contained in [3], where multiplier techniques are used.

3.2. *Sharpness of the Converse Estimate*

We would like to conclude the paper by showing that the Dini condition (3.3) in Theorem 3.1, part B, cannot be weakened. To this end we need two further properties of the K-functional with respect to fractional derivatives. (The following can also be proved by multiplier techniques, see, e.g., [26].)

LEMMA 3.9. *Let $f \in H^p$, $0 < p < \infty$, $\alpha, \beta > 0$ and $\delta \in (0, 1)$. Then*

$$K_{\alpha+\beta}(\delta^{\alpha+\beta}, f)_p \lesssim K_{\alpha}(\delta^{\alpha}, f)_p.$$

Proof. Let $g^{(\alpha)} \in H^p$. Then it is clear that

$$K_{\alpha+\beta}(\delta^{\alpha+\beta}, f)_p \lesssim \|f-g\|_p + K_{\alpha+\beta}(\delta^{\alpha+\beta}, g)_p.$$

By Lemma 3.7 and Lemma 3.6 we obtain the assertion ($s = \min(1, p)$)

$$K_{\alpha+\beta}(\delta^{\alpha+\beta}, g)_p^s \lesssim \int_{1-\delta}^1 (1-\rho)^{(\alpha+\beta)\, s-1}\, \|g^{(\alpha+\beta)}(\rho e^{it})\|_p^s\, d\rho$$

$$\lesssim \int_{1-\delta}^1 (1-\rho)^{\alpha s-1}\, \|g^{(\alpha)}(e^{it})\|_p^s\, d\rho$$

$$\lesssim \delta^{\alpha s}\, \|g^{(\alpha)}\|_p^s.$$

Next we give a characterization of all K-functionals in $H^p(\mathbb{D})$, $p > 0$.

Let $\alpha > 0$. Denote by W_α the class of all nonnegative, continuous, bounded functions on $(0, \infty)$ such that

(1) $\omega(t) \to 0, \quad t \to 0+$;
(2) $\omega(t)$ is nondecreasing;
(3) $t^{-\alpha}\omega(t)$ is nonincreasing.

LEMMA 3.10. *Let $\alpha > 0$ and $p > 0$.*

(A) *If $\omega \in W_\alpha$, then there is a function $f \in H^p$ such that*

$$K_\alpha(\delta^\alpha, f)_p \approx \omega(\delta). \tag{3.19}$$

(B) *Conversely, for any $\alpha > 0$ and $f \in H^p$ there is some $\omega \in W_\alpha$ such that* (3.19) *is true.*

Proof. Fix α, p and $\omega \in W_\alpha$; following Oskolkov [16] put for $i = 0, 1, 2, \ldots$

$$m_0 = [\alpha] + 1, \qquad m_{i+1} = \min\left\{ m \in \mathbb{N} : \max\left(\frac{\omega(2^{-m})}{\omega(2^{-m_i})}, \frac{2^{m_i\alpha}\omega(2^{-m_i})}{2^{m\alpha}\omega(2^{-m})} \right) \leqslant \frac{1}{2} \right\},$$

and set

$$f(z) = \sum_{k=0}^{\infty} \omega(n_k^{-1})\, z^{n_k}, \qquad n_i = 2^{m_i}.$$

Let $\delta \in [n_{i+1}^{-1}, n_i^{-1})$ and $P_{n_i}(z) = \sum_{k=1}^{i} \omega(n_k^{-1})\, z^{n_k}$. Then

$$K_\alpha(\delta^\alpha, f)_p \approx \|f - P_{n_i}\|_p + \delta^\alpha \|P_{n_i}^{(\alpha)}\|_p.$$

The $\lesssim$-direction is obvious. For the converse we use a property of lacunary series (see [29, p. 215]), giving

$$\|f - P_{n_i}\|_p \approx \left(\sum_{k=i+1}^{\infty} \omega^2(n_k^{-1}) \right)^{1/2} \lesssim \|f - R\|_p,$$

where the polynomial R of degree $n_{i+1} - 1$ is that from Theorem 2.3. From this and Lemma 3.9

$$\|f - P_{n_i}\|_p \lesssim K_m((n_{i+1} - 1)^{-1}, f)_p \lesssim K_\alpha(\delta^\alpha, f)_p, \qquad m = [\alpha] + 1, \tag{3.20}$$

and, by Lemma C and Theorem 3.1,

$$\delta^\alpha \|P_{n_i}^{(\alpha)}\|_p \lesssim \delta^\alpha \|P_{n_i}^{(\alpha)}((1-n_i^{-1})\, e^{it})\|_p \lesssim \delta^\alpha \|P_{n_i}^{(\alpha)}((1-\delta)\, e^{it})\|_p$$
$$\lesssim K_\alpha(\delta^\alpha, P_{n_i}-f)+K_\alpha(\delta^\alpha, f) \lesssim K_\alpha(\delta^\alpha, f)_p.$$

So

$$K_\alpha(\delta^\alpha, f)_p \approx \left\| \sum_{k=i+1}^{\infty} \omega(n_k^{-1})\, z^{n_k} \right\|_p + \delta^\alpha \left\| \sum_{k=0}^{i} n_k^\alpha \omega(n_k^{-1})\, z^{n_k} \right\|_p$$
$$\approx \omega(n_{i+1}^{-1}) + \delta^\alpha n_i^\alpha \omega(n_i^{-1}) \approx \omega(\delta) \left\{ \frac{\omega(n_{i+1}^{-1})}{\omega(\delta)} + \frac{n_i^\alpha \omega(n_i^{-1})}{\delta^{-\alpha}\omega(\delta)} \right\} \approx \omega(\delta).$$

To prove part (B) note that $K_\alpha(\delta^\alpha, f)_p$ is a concave function with respect to δ^α and thus, by [4, Lemma 3.1.1], belongs to the class W_α.

We are now able to prove that Theorem 3.1 is sharp.

THEOREM 3.11. *Let $\omega \in W_\alpha$, $\alpha > 0$, and let*

$$\limsup_{\delta \to 0+} \omega^{-1}(\delta) \int_0^\delta \omega(t)\, t^{-1}\, dt = +\infty.$$

Then there is an analytic function $f(z)$, such that

$$\|f^{(\alpha)}(re^{i\varphi})\|_p \lesssim (1-r)^{-\alpha}\, \omega(1-r), \qquad 0<r<1,$$

but

$$\limsup_{\delta \to 0+} \omega^{-1}(\delta)\, K_\alpha(\delta^\alpha, f) = +\infty.$$

Proof. Write

$$f(z) = \sum_{k=0}^{\infty} \omega(n_k^{-1})\, z^{n_k}, \qquad n_k = 2^{i_k},$$

$$i_0 = 0, \qquad i_{k+1} = \min\left\{ i\colon \frac{2^{i_k\alpha}\omega(2^{-i_k})}{2^{i\alpha}\omega(2^{-i})} \leqslant 2^{-1/s} \right\}, \qquad s = \min(1, p).$$

We shall show that

(A) $\|f^{(\alpha)}(re^{i\varphi})\|_p \lesssim (1-r)^{-\alpha}\, \omega(1-r)$;

(B) $\omega^{-1}(n_k^{-1})\, K_\alpha(n_k^{-\alpha}, f) \gtrsim \left(\omega^{-2}(n_k^{-1}) \sum_{j=k+1}^{\infty} \omega^2(n_j^{-1}) \right)^{1/2}, \qquad n_k \geqslant \alpha;$

(C) for some subsequence $\{s_k\}$ of $\{n_k\}$ there holds

$$\lim_{k\to\infty} \omega^{-1/\alpha}(s_k^{-1}) \sum_{n_j \geqslant s_k} \omega^{1/\alpha}(n_j^{-1}) = \infty.$$

The properties (A)–(C) are sufficient to prove Theorem 3.2, since the inequalities

$$\sum_{j=k}^{\infty} a_j^{1/\alpha} \lesssim a_k^{1/\alpha} \qquad \text{and} \qquad \sum_{j=k}^{\infty} a_j^2 \lesssim a_k^2, \qquad a_k \downarrow 0,$$

are equivalent (see Remark 3.2 (ii)).

Proof of (A). First, estimate $\|f^{(\alpha)}(r_j e^{i\varphi})\|_p^s$ for $r_j = 1 - n_j^{-1}$. To this end note, if $\gamma_k = n_k^{\alpha s} \omega^s(n_k^{-1})$, then

$$2^{-(1+\alpha s)} \leqslant \gamma_k / \gamma_{k+1} \leqslant 2^{-1}. \tag{3.21}$$

For any $g(z) = \sum_{k=0}^{\infty} a_k z^k$ the following inequality is true (see (3.12))

$$\|g^{(\alpha)}(z)\|_{H^p} \lesssim \left\| \sum_{k=[\alpha]}^{\infty} (k+1)^\alpha \, a_k z^{k-[\alpha]} \right\|_{H^p}$$

and, therefore,

$$\|f^{(\alpha)}(r_j e^{i\varphi})\|_p^s \lesssim \sum_{k=0}^{j-1} \gamma_k + \sum_{k=j}^{\infty} r_j^{n_k} \gamma_k \lesssim \gamma_{j-1} + \sum_{k=j}^{\infty} \exp\left(-\frac{n_k}{n_j} s\right) \gamma_k \lesssim \gamma_j.$$

Now choose $r_{j-1} < r \leqslant r_j$. The relation (3.21), the monotonicity of $\|f^{(\alpha)}(re^{i\varphi})\|_p$ and of $\omega(1-r)/(1-r)^\alpha$ as functions of r give

$$\|f^{(\alpha)}(re^{i\varphi})\|_p^s \lesssim \|f^{(\alpha)}(r_j e^{i\varphi})\|_p^s \lesssim \gamma_j \lesssim \gamma_{j-1} \lesssim (1-r)^{-\alpha s} \, \omega^s(1-r).$$

The assertion (B) is already deduced in the proof of Lemma 3.10—see (3.20).

Proof of (C). Let

$$\lim_{k \to \infty} \left\{ \int_0^{\delta_k} \frac{\omega^{1/\alpha}(t)}{t} \, dt \right\} \cdot \omega^{-1/\alpha}(\delta_k) = \infty.$$

By making $\{\delta_k\}$ less dense if needed, we can obtain such a situation that each interval $[n_k^{-1}, n_{k-1}^{-1})$ contains not more than one δ_k. So we may assume that there exists a strictly increasing function h with $h(k) \in \mathbb{N}$ such that the intervals $\Delta_k := [n_{h(k)}^{-1}, n_{h(k)-1}^{-1})$ contain one δ_k. The monotonicity of $\omega^{1/\alpha}(t) \cdot t^{-1}$ and the choice of n_k give

$$\frac{\omega^{1/\alpha}(\delta)}{\delta} \cdot \frac{t}{\omega^{1/\alpha}(t)} \approx 1, \qquad t, \delta \in \Delta_k.$$

Using the notation $s_k = n_{h(k)}$ it is clear that $\omega^{-1/\alpha}(s_k^{-1}) \geqslant \omega^{-1/\alpha}(\delta_k)$. Therefore,

$$\omega^{-1/\alpha}(s_k^{-1}) \int_0^{s_k^{-1}} \frac{\omega^{1/\alpha}(t)}{t}\,dt$$
$$\gtrsim \omega^{-1/\alpha}(\delta_k) \left(\int_0^{\delta_k} - \int_{s_k^{-1}}^{\delta_k} \right) \frac{\omega^{1/\alpha}(t)}{t}\,dt$$
$$\gtrsim \omega^{-1/\alpha}(\delta_k) \int_0^{\delta_k} \frac{\omega^{1/\alpha}(t)}{t}\,dt - (\delta_k - s_k^{-1})\, \omega^{-1/\alpha}(\delta_k)\, \frac{\omega^{1/\alpha}(\delta_k)}{\delta_k}.$$

So we have

$$\lim_{k\to\infty} \omega^{-1/\alpha}(s_k^{-1}) \int_0^{s_k^{-1}} \frac{\omega^{1/\alpha}(t)}{t}\,dt = \infty,$$

or

$$\lim_{k\to\infty} \omega^{-1/\alpha}(s_k^{-1}) \sum_{n_j \geqslant s_k} \omega^{1/\alpha}(n_j^{-1}) = \infty.$$

REFERENCES

1. G. E. Andrews, "The Theory of Partitions," Addison–Wesley, Reading, MA, 1976.
2. E. S. Belinskii, Strong summability of Fourier series of the periodic functions from $H^p (0 < p \leqslant 1)$, *Constr. Approx.* **12** (1996), 187–195.
3. E. S. Belinskii, Strong summability for the Marcinkiewicz means in the integral metric and related questions, *J. Austral. Math. Soc. Ser. A* **65** (1998), 303–312.
4. J. Bergh and J. Löfström, "Interpolation Spaces: An Introduction," Springer-Verlag, Berlin, 1976.
5. Yu. A. Brudnyi and E. A. Gopengauz, Generalization of a theorem of Hardy and Littlewood, *Mat. Sb.* **52** (1960), 891–894. [In Russian]
6. R. A. DeVore and G. G. Lorentz, "Constructive Approximation," Springer-Verlag, Berlin, 1993.
7. P. L. Duren, "Theory of H^p-Spaces," Academic Press, New York, 1970.
8. T. M. Flett, Lipschitz spaces of functions on the circle and the disc, *J. Math. Anal. Appl.* **39** (1972), 125–158.
9. J. B. Garnett, "Bounded Analytic Functions," Academic Press, New York, 1981.
10. T. N. T. Goodman, H. Oruç, and G. M. Phillips, Convexity and generalized Bernstein polynomials, *Proc. Edinburgh Math. Soc. (2)* **42** (1999), 179–190.
11. G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals I, *Math. Z.* **27** (1928), 565–606.
12. G. H. Hardy and J. E. Littlewood, A maximal theorem with function-theoretic applications, *Acta Math.* **54** (1930), 81–116.
13. G. H. Hardy and J. E. Littlewood, Some properties of fractional integrals, II, *Math. Z.* **34** (1932), 403–439.
14. E. Hille, G. Szegö, and J. D. Tamarkin, On some generalizations of a theorem of A. Markoff, *Duke Math. J.* **3** (1937), 729–739.

15. Yu. V. Kryakin, On a theorem of Hardy and Littlewood, *Math. Notes* **53** (1993), 167–171.
16. K. I. Oskolkov, Approximation properties of summable functions on sets of full measure, *Math. USSR-Sb.* **32** (1977), 489–514.
17. P. Oswald, On some approximation properties of real Hardy spaces $(0 < p \leqslant 1)$, *J. Approx. Theory* **40** (1984), 45–65.
18. A. A. Pekarskii, Inequalities of Bernstein type for derivatives of rational functions, and inverse theorems of rational approximation, *Math. USSR-Sb.* **52** (1985), 557–574.
19. F. Riesz, Über die Randwerte einer analytischen Funktion, *Math. Z.* **18** (1923), 87–95.
20. S. G. Samko, A. A. Kilbas, and O. I. Marichev, "Fractional Integrals and Derivatives: Theory and Applications," Gordon & Breach, Yverdon, 1993.
21. A. A. Soljanik, On the order of approximation to functions of $H^p(R)$ $(0 < p \leqslant 1)$ by certain means of Fourier integrals, *Anal. Math.* **12** (1986), 59–76.
22. E. A. Storozhenko, Theorems of Jackson type in H^p, $0 < p < 1$, *Math. USSR Izv.* **17** (1981), 203–217.
23. E. A. Storozhenko, On a problem of Hardy and Littlewood, *Math. USSR-Sb.* **47** (1984), 557–577.
24. E. A. Storozhenko and V. G. Krotov, Approximate and differential-difference properties of functions in Hardy spaces H^p, $0 < p < 1$, *in* "Theory of Functions and Approximations," Part 1, pp. 65–81, Saratov. Gos. Univ., Saratov, 1983. [In Russian]
25. P. M. Tamrazov, "Smoothnesses and Polynomial Approximations," Naukova Dumka, Moscow, 1975. [In Russian]
26. W. Trebels, Equivalence of a K-functional with the approximation behavior of some linear means for abstract Fourier series, *Proc. Amer. Math. Soc.* **127** (1999), 2883–2887.
27. Ya. Valashek, "On Approximation in Hardy Classes in Polydiscs," Kandidat Dissert. Fiz.-Mat. Nauk, Odessa, 1982. [In Russian]
28. A. Zygmund, Smooth functions, *Duke Math. J.* **12** (1945), 47–76.
29. A. Zygmund, "Trigonometric Series," Vol. 1, Cambridge Univ. Press, Cambridge, UK, 1959.